THE JOURNAL OF SYMBOLIC LOGIC
Volume 25, Number 2, June 1960

THE SENTENTIAL CALCULUS USING RULE OF INFERENCE R_e

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Axiomatizations of the sentential calculus which use $\mathbf{R}_{mp}$ (*modus ponens*), have been shown equivalent to axiomatizations similar in all respects except that $\mathbf{R}_{mp}$ is replaced by the less restricted rule $\mathbf{R}_e$ (*rule of excision*)[1]:

> $\mathbf{R}_e$. *If* S *and* (... (S⊃S′) ...), *then* (... S′ ...).
> (Read: "If a formula 'S' and any formula 'T' containing '(S⊃S′)' as a component are given, then one may proceed to a new formula 'T″' got by substituting 'S″' for '(S⊃S′)' in 'T'.")

We show here that $\mathbf{R}_e$ can be more powerful than $\mathbf{R}_{mp}$, hence is independent.

I. The two following axioms, with $\mathbf{R}_e$, *yield the sentential calculus*:

A1. $((s\supset(p\supset q))\supset((s\supset(q\supset r))\supset(s\supset(p\supset r))))$

A2. $(r\supset((-q\supset -p)\supset(p\supset q)))$

For, from these, with $\mathbf{R}_e$, we derive *1, *10, and *11 below; these constitute an axiom set which, with $\mathbf{R}_{mp}$, yields the full sentential calculus[2]:

*1. $((-q\supset -p)\supset(p\supset q))$ [A2, A2 $r/(r\supset((-q\supset -p)\supset(p\supset q)))$, $\mathbf{R}_e$]

*2. $((p\supset q)\supset((q\supset r)\supset(p\supset r)))$ [*1, A1 $s/((-q\supset -p)\supset(p\supset q))$, $\mathbf{R}_e$]

*3. $(q\supset((q\supset r)\supset r))$ [*1, *2 $p/((-q\supset -p)\supset(p\supset q))$, $\mathbf{R}_e$]

*4. $(-p\supset -p)$ [*1, *3 $q/((-q\supset -p)\supset(p\supset q))$, $r/-p$, $\mathbf{R}_e$]

*5. $((s\supset p)\supset((s\supset(p\supset q))\supset(s\supset q)))$ [*4, A1 $p/(-p\supset -p)$, q/p, r/q, $\mathbf{R}_e$]

*6. $(q\supset(p\supset p))$ [*4, A2 r/q, q/p, $\mathbf{R}_e$]

*7. $((((q\supset r)\supset r)\supset(p\supset r))\supset(q\supset(p\supset r)))$ [*3, *2 p/q, $q/((q\supset r)\supset r)$, $r/(p\supset r)$, $\mathbf{R}_e$]

*8. $((p\supset(q\supset r))\supset(((q\supset r)\supset r)\supset(p\supset r)))$ [*2 $q/(q\supset r)$]

*9. $((p\supset(q\supset r))\supset(q\supset(p\supset r)))$ [*7, *8, *2 $p/(p\supset(q\supset r))$, $q/(((q\supset r)\supset r)\supset(p\supset r))$, $r/(q\supset(p\supset r))$, $\mathbf{R}_e$]

*10. $(p\supset(q\supset p))$ [*6, *9 p/q, q/p, r/p, $\mathbf{R}_e$]

*11. $((s\supset(p\supset q))\supset((s\supset p)\supset(s\supset q)))$ [*5, *9 $p/(s\supset p)$, $q/(s\supset(p\supset q))$, $r/(s\supset q)$, $\mathbf{R}_e$]

II. It is not possible to obtain '$(p\supset p)$' *from* A1 *and* A2 *if* $\mathbf{R}_{mp}$ *replaces* $\mathbf{R}_e$.

This is shown by assigning the accompanying interpretations, on which all provable theorems take only the value '0', to the primitive symbols '⊃' and '—'.

⊃	0	1	2
0	0	2	2
1	0	2	2
2	0	0	0

—	p
2	0
1	1
0	2

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Received December 7, 1959.

[1] R. B. ANGELL, *On a less restricted type of rule of inference*, **Mind**, n.s. vol. LXIX (1960).

[2] J. ŁUKASIEWICZ and A. TARSKI, *Untersuchungen über den Aussagenkalkül*, **Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie**, Classe III, vol. 23 (1930), pp. 30–50.